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# THE STABILITY OF A "SLEEPING" LAGRANGE TOP WITH A VIBRATING SUSPENSION POINT†

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The motion of a Lagrange top, the suspension point of which performs vertical harmonic oscillations of arbitrary frequency and amplitude, is considered. The particular motion where the top rotates about a vertically positioned axis of symmetry at a constant angular velocity (a "sleeping" top) is investigated. The complete solution of the problem concerning the stability of such motion for all permissible values of the parameters of the problem is given. © 2001 Elsevier Science Ltd. All rights reserved.

Some aspects of the motion of a Lagrange top in the case of vertical harmonic oscillations of the suspension point were considered earlier [1].

## 1. FORMULATION OF THE PROBLEM

Consider the motion of a Lagrange top (a dynamically symmetrical rigid body whose centre of mass lies on the axis of symmetry) about its suspension point $O$. We will assume that the point $O$ performs vertical harmonic oscillations according to the law $\xi(t) = a_* \cos \Omega t$ about a certain fixed point.

We will introduce a translating reference frame $OXYZ$ (the $OZ$ axis is directed vertically upwards) and a reference frame $Oxyz$ connected with the top, the axes of which coincide with the main axes of inertial of the top for the point $O$, where the $Oz$ axis is directed along its axis of dynamic symmetry, and the centre of mass $G$ lies on the positive half-axis $Oz$ ($OG = z_G$, $z_G > 0$). We will specify the orientation of the reference frame $Oxyz$ with respect to $OXYZ$ using the Euler angles.

The kinetic and potential energies of the top are calculated from the formulae [1]

$$T = \frac{1}{2}m\dot{\xi}^2 - mz_G\dot{\xi}\dot{\theta}\sin\theta + \frac{1}{2}A(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}C(\dot{\psi}\cos\theta + \dot{\varphi})^2$$
$$\Pi = mgz_G\cos\theta + mg\xi(t) \tag{1.1}$$

where $m$ is the mass of the top, and $A$ and $C$ are the equatorial and axial moments of inertia, respectively.

The coordinates $\psi$ and $\varphi$ are cyclic; denoting the constant values of the momenta $p_\psi$ and $p_\varphi$ by $A\Omega a$ and $A\Omega b$ respectively ($a$ and $b$ are dimensionless constants), we obtain from (1.1) the following expressions for the angular velocities of precession and natural rotation of the top

$$\psi' = \frac{a - b\cos\theta}{\sin^2\theta}, \quad \varphi' = \frac{A}{C}b - \frac{(a - b\cos\theta)\cos\theta}{\sin^2\theta}$$

where the prime denotes differentiation with respect to the variable $\tau = \Omega t$.

The investigation of the motion of the top reduces to considering a system with one degree of freedom with the generalized coordinate $\theta$; the reduced Hamiltonian has the form [1]

$$H = \frac{(a - b\cos\theta)^2}{2\sin^2\theta} + \frac{1}{2}(p_\theta - \beta\sin\tau\sin\theta)^2 + d\cos\theta \tag{1.2}$$

In (1.2), $p_\theta$ is the momentum (dimensionless) corresponding to the coordinate $\theta$, while the parameters $\beta$ and $d$ are defined by the formulae

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$$\beta = \frac{a_*}{l_0}, \quad d = \frac{mgz_G}{A\Omega^2}$$

where $l_0 = A/(mz_G)$ is the reduced length of the body as a physical pendulum. The quantity $\beta$ ($\beta \geqslant 0$) characterizes the amplitude of the oscillations of the suspension point, while the parameter $d$ ($d > 0$) characterizes the position of the centre of mass of the top on the axis of symmetry.

Suppose the values of the constants $a$ and $b$ of the cyclic integrals are related as follows: $|a| = |b| \neq 0$. When $a = b$, Hamiltonian (1.2) and the differential equation of motion have the form

$$H = \frac{1}{2}\zeta \operatorname{tg}^2\frac{\theta}{2} + \frac{1}{2}(p_\theta - \beta\sin\tau\sin\theta)^2 + d\cos\theta \quad (\zeta = a^2) \tag{1.3}$$

$$\theta'' + \frac{\zeta \operatorname{tg}(\theta/2)}{2\cos^2(\theta/2)} + (-d + \beta\cos\tau)\sin\theta = 0 \tag{1.4}$$

When $a = -b$, the Hamiltonian is obtained from (1.3) by replacing $tg^2(\theta/2)$ by $ctg^2(\theta/2)$, while the equation of motion is obtained from (1.4) by replacing $tg(\theta/2)/\cos^2(\theta/2)$ by $-ctg(\theta/2)/\sin^2(\theta/2)$.

Equation (1.4) has the particular solution (the position of equilibrium) $\theta = 0$, and the corresponding equation when $a = -b$ has the solution $\theta = \pi$, which correspond to motion where the top rotates about a vertically positioned axis of symmetry at a constant angular velocity; when $a = b$, the centre of mass of the top lies above the suspension point an "inverted" top), and when $a = -b$ it lies below this point ("hanging" top).

When the suspension point does not evaluate, we have the classical "sleeping" Lagrange top; the "hanging" top is stable, and the condition of stability of the "inverted" top is the well-known Maiyevskii–Chetayev condition $C^2\omega^2 \geqslant 4Amgz_G$ ($\omega$ is the angular velocity of rotation of the top about the axis of symmetry), which in our notation has the form.

$$\zeta/4 \geqslant d \tag{1.5}$$

The purpose of the present paper is to solve the problem of the stability of a "sleeping" Lagrange top (with respect to the variables $\theta$ and $p_\theta$) when there are vertical harmonic oscillations of the suspension point of arbitrary frequency and amplitude.

The case when $|a| = |b| = 0$ corresponds to motion of the top as a physical pendulum. The problem of the stability of the relative equilibria of a mathematical pendulum on the vertical in the case of vertical harmonic oscillations of its suspension point was solved earlier [2].

Putting $\theta = q$ in (1.4) and $\theta = \pi + q$ in the corresponding equation for the case when $a = -b$, we obtain the equations of perturbed motion. If, in the second of these equations, $\tau$ is replaced by $\tau + \pi$, and $d$ is replaced by $-d$, the equations of perturbed motion (for the cases $a = b$ and $a = -b$) are vertical. Therefore, it is sufficient to consider Eq (1.4) and the stability of its equilibria $\theta = 0$, assuming the $\beta > 0$ and $-\infty < d < +\infty$. When $\beta > 0$ and $d > 0$, the conclusions will relate to the "inverted" top ($a = b$), and when $\beta > 0$ and $d < 0$, they will relate to the "hanging" top ($a = -b$).

In (1.3) we carry out a canonical univalent replacement of the variables $0, p_\theta \to q, p$ by means of the formulae $q = \theta$ and $p = p_\theta - \beta\sin\tau\sin\theta$. The Hamiltonian then becomes

$$H = \frac{1}{2}p^2 + (d - \beta\cos\tau)\cos q + \frac{1}{2}\zeta \operatorname{tg}^2\frac{q}{2} \tag{1.6}$$

In the vicinity of the equilibrium position $q = 0, p = 0$, we expand expression (1.6) in series in powers of $q$ and $p$

$$H = H_2 + H_4 + \ldots \tag{1.7}$$

$$H_2 = \frac{1}{2}p^2 + \frac{1}{2}(\alpha + \beta\cos\tau)q^2, \quad H_4 = -\frac{1}{24}\left(\alpha - \frac{3}{4}\zeta + \beta\cos\tau\right)q^4; \quad \alpha = \frac{1}{4}\zeta - d$$

Since $\xi > 0$, then, for the "hanging" top ($d < 0$) the parameter $\alpha$ is always positive; for the "inverted" top ($d > 0$), however, $-\infty < \alpha < +\infty$, and here, according to (1.5), the inequality $\alpha \geqslant 0$ is the condition of stability of such a top in the classical case. We will consider the problem of the stability of the equilibrium position $q = 0, p = 0$ of a system with Hamiltonian (1.7), for all permissible values of the parameters $\alpha$, $\beta$ and $\xi$. Then, using the last relation of (1.7), it is possible, instead of $\xi$, to change to the parameter $d$ and to interpret the conclusions obtained concerning the stability as it applies to the "inverted' top ($d > 0$) or the "hanging" top ($d < 0$).

## 2. STABILITY IN THE LINEAR APPROXIMATION

The linearized equations of motion of the system with Hamiltonian (1.7) reduce to a Mathieu equation of the form

$$q'' + (\alpha + \beta\cos\tau)q = 0 \tag{2.1}$$

The problem of the stability of the solution $q = 0$ of Eq. (2.1) has been conjugated in detail; brief information [3–5] that will subsequently be needed will be given.

Let $\mathbf{X}(\tau) = \| x_{ij}(\tau) \|_{i,j=1}^{2}$ $(x_{21} = x'_{11}, x_{22} = x'_{12})$ be the fundamental matrix of solutions of the linearized system, described by the Hamiltonian $H_2$ [see (1.7)], which satisfies the condition $\mathbf{X}(0) = \mathbf{E}$, where $\mathbf{E}$ is the identity matrix. The functions $x_{11}(\tau)$ and $x_{22}(\tau)$ are even, while $x_{12}(\tau)$ and $x_{21}(\tau)$ are uneven with respect to $\tau$. The characteristic equation of the linearized system has the form

$$\rho^2 - 2A\rho + 1 = 0, \quad A = x_{11}(2\pi) = x_{22}(2\pi) \tag{2.2}$$

Figure 1, in the plane of the parameters $(\alpha, \beta)$ $(-\infty < \alpha < +\infty, \beta \geqslant 0)$, shows the regions of stability and instability of the zero solution of Eq. (2.1). In the shaded regions, one of the roots of the characteristic equation (2.2) is greater than unity in modulus, and we have instability (not only in the linear problem but also for the complete system of equations of perturbed motion, which follows from Lyapunov's theorem of stability in the first approximation [6]).

In the unshaded regions, the roots of Eq. (2.2) are complex-conjugate and have moduli equal to unity, i.e. the conditions of stability in the linear approximation are satisfied. We will denote by $g_n$ $(n = 1, 2, \ldots)$ the region of stability which, as $\beta \to 0$, passes into the interval $(n-1)^2/4 < \alpha < n^2/4$ of the $\beta$ axis. The characteristic indices $\pm i\lambda(\lambda > 0)$ in the regions $g_n$ are purely imaginary, and the quantity $\lambda$ is determined from the relation $\cos 2\pi\lambda = A(|A| < 1)$.

All regions of stability intersect the $\alpha = 0$ axis and, as $\beta$ increases, become very narrow; the slope of the curves bounding them approaches $-1$ as $\beta \to +\infty$.

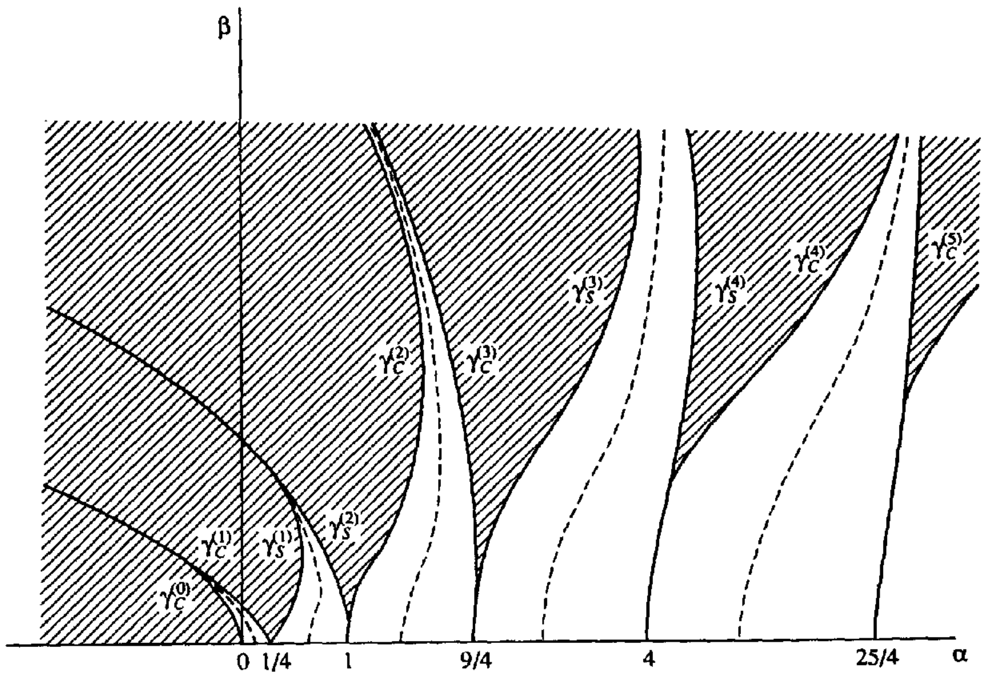


Fig. 1

The curvilinear boundaries of the regions of stability and instability emanate in pairs from the points $\alpha = n^2/4$ ($n = 1, 2, \ldots$) of the axis $\beta = 0$; for small $\beta$, these pairs of curves have a tangent of the order of $n - 1$. We will use $\gamma_C^{(2k-2)}$, $\gamma_C^{(2k-1)}$ and $\gamma^{(2k-1)}$, $\gamma_S^{(2k)}$ to denote the boundaries of the regions $g_{2k-1}$ and $g_{2k}$ ($k = 1, 2, \ldots$), respectively. On the curves $\gamma_C^{(0)}$, $\gamma_C^{(2k)}$ and $\gamma_S^{(2k)}$($k = 1, 2, \ldots$) we have first-order resonance (the roots $\rho_1$ and $\rho_2$ of Eq. (2.1) are equal to unity), and on the curves $\gamma_C^{(2k-1)}$ and $\gamma_S^{(2k-1)}$ ($k = 1, 2, \ldots$) we have second-order resonance ($\rho_1 = \rho_2 = -1$).

The elements of the fundamental matrix of the linearized system of equations of perturbed motion for points of the boundary curves $\gamma_C^{(m)}$ ($m = 0, 1, 2, \ldots$) are determined by an even Mathieu function of the first kind and by the corresponding Mathieu function of the second kind and for points of the curves $\gamma_S^{(m)}$ ($m = 1, 2, \ldots$) by an odd Mathieu function of the first kind and the corresponding Mathieu function of the second kind. The Mathieu functions of the first kind are $2\pi$-periodic in $\tau$ for even values of $m$, and $4\pi$-periodic in $\tau$ for odd values of $m$, while the Mathieu functions of the second kind are non-periodic and unbounded. Therefore, on the boundary curves, the overall solution of the linearized system of equations is unbounded, and the equilibrium position considered is unstable in the linear approximation.

The following sections give a rigorous solution of the problem of stability within the regions $g_n$ of stability in the linear approximation and on the boundary curves. Methods and algorithms developed in previous studies [7–10] are used, as well as certain relations obtained [2] by solving the problem of the stability of the relative equilibrium positions of a pendulum with vertical oscillations of the suspension point.

## 3. ANALYSIS OF STABILITY IN THE REGIONS $G_N$. THE NON-RESONANT CASE

We will first consider the problem of the stability of a system with Hamiltonian (1.7) for $\alpha$ and $\beta$ values within the regions $g_n$ of stability in the linear approximation. For this, using a series of canonical transformations, we will reduce the Hamiltonian of the perturbed motion to the normal form.

Using the linear canonical replacement, $2\pi$-periodic in $\tau$, of the variables $q, p \to q_*, p_*$ of the form [9].

$$\begin{gathered} q = n_{11}(\tau)q_* + n_{12}(\tau)p_*, \qquad q = n_{21}(\tau)q_* + n_{22}(\tau)p_* \\ n_{i1}(\tau) = \varkappa^{-1/2}(\mu_i \cos\lambda\tau + \nu_i \sin\lambda\tau), \quad n_{i2}(\tau) = \varkappa^{-1/2}(-\mu_i \sin\lambda\tau + \nu_i \cos\lambda\tau) \\ \mu_i = \sin 2\pi\lambda x_{i2}(\tau), \quad \nu_i = -x_{12}(2\pi)x_{i1}(\tau), \quad i = 1, 2 \\ \varkappa = x_{12}(2\pi)\sin 2\pi\lambda > 0 \end{gathered} \tag{3.1}$$

the quadratic part of Hamiltonian (1.7) is transformed into the normal form $\lambda(q_*^2 + p_*^2)/2$, and the Hamiltonian will take the form

$$H = \frac{1}{2}\lambda(q_*^2 + p_*^2) - \frac{1}{24}\left(\alpha - \frac{3}{4}\zeta + \beta\cos\tau\right)(n_{11}q_* + n_{12}p_*)^4 + O_6 \tag{3.2}$$

where $O_6$ is a set of terms no lower than the sixth power in $q_*$ and $p_*$.

The quantity $\lambda$ is determined from the relation $\cos 2\pi\lambda = A$; to eliminate the ambiguity of this definition, we will assume that $\lambda = \sqrt{\alpha}$ when $\beta = 0$, and take into account the property of continuity of the characteristic exponents with respect to $\beta$. Then, we obtain that $\lambda = (2\pi)^{-1}\arccos A + k - 1$ in the regions $g_{2k-1}$ and that $\lambda = -(2\pi)^{-1}\arccos A + k$ in the regions $g_{2k}$ ($k = 1, 2, \ldots$).

Suppose that, to begin with, fourth-order resonance does not occur in the system, i.e. the quantity $4\lambda$ is not an integer. A fourth-order resonance curve does exist, and only one, in each of the regions $g_n$ ($n = 1, 2, \ldots$). On this curve, $4\lambda = 2n - 1$, and it emanates from the point $\alpha_0 = (2n - 1)^2/16$ of the $\beta = 0$ axis for small $\beta$ it is given by the equation

$$\alpha = \alpha_0 + \beta^2/(2(4\alpha_0 - 1)) + O(\beta^4)$$

and, as $\beta$ increases it extends without limit into the regions $g_n$. The fourth-order resonance curves are shown in Fig. 1 by the dashed lines.

Using a canonical replacement of the variables $q_*, p_* \to x, y$ of the Birkhoff transformation type that is near-identical, real, $2\pi$-periodic in $\tau$, and analytical in $x, y$, Hamiltonian (3.2) can be reduced to the form

$$H = \frac{1}{2}\lambda(x^2+y^2)+\frac{1}{4}c_2(x^2+y^2)^2+O_6 \tag{3.3}$$

where the constant coefficient $c_2$ is calculated by means of the formula [9] (below, with the exception of Section 5, integration with respect to $\tau$ is always carried out over the range from 0 to $2\pi$)

$$c_2 = -\frac{1}{32\pi}\int\left(\alpha-\frac{3}{4}\zeta+\beta\cos\tau\right)(n_{11}^2+n_{12}^2)^2 d\tau \tag{3.4}$$

If $c_2 \neq 0$, then, by the Arnold–Moser theorem [7, 8], we have stability.

Taking into account that the functions $\mu_1(\tau)$ and $\nu_1(\tau)$ occurring in the expressions for $n_{11}$ and $n_{12}$ (see (3.1)) satisfy Mathieu's equation (2.1), by transforming the right-hand rule of (3.4) we obtain

$$c_2 = -\frac{1}{32\pi\varkappa^2}\int\left[(\mu_1'\nu_1+\mu_1\nu_1')^2+(\mu_1\mu_1'-\nu_1\nu_1')^2+2(\mu_1\mu_1'+\nu_1\nu_1')^2-\frac{3}{4}\zeta(\mu_1^2+\nu_1^2)^2\right]d\tau \tag{3.5}$$

For fixed values of the parameters $\alpha$ and $\beta$, the coefficient $c_2$ vanishes if

$$\zeta = \zeta_*(\alpha,\beta) = \frac{4}{3}\frac{\int[(\mu_1'\nu_1+\mu_1\nu_1')^2+(\mu_1\mu_1'-\nu_1\nu_1')^2+2(\mu_1\mu_1'+\nu_1\nu_1')^2]d\tau}{\int(\mu_1^2+\nu_1^2)^2 d\tau} \tag{3.6}$$

Thus, each point $(\alpha, \beta)$ belonging to one of the regions $g_n$ of stability in the linear approximation (besides the points of the resonance curves) has its corresponding unique point $\xi = \xi_*(\alpha, \beta)$ where the condition of non-degeneracy $c_2 \neq 0$ of Hamiltonian (3.3) is violated.

We will first consider the case when $\beta \ll 1$ (the amplitude of oscillations of the suspension point of the top is small). Following the algorithm from [4], we can write the elements of the fundamental matrix $X(\tau)$ for points of the curve $\alpha = \alpha_0 + \beta^2/(2(4\alpha_0 - 1)) + O(\beta^4)$ (emanating from the point $\alpha = \alpha_0$ ($\alpha_0 \neq k^2/4$, $k = 0, 1, 2, \ldots$) of the $\beta = 0$ axis), on which $A = \cos 2\pi\sqrt{\alpha_0}$. Then, carrying out a series of transformations using formulae (3.1) and (3.6), we obtain that, in the case of small $\beta$, the condition of non-degeneracy is violated when

$$\zeta = \zeta_* = \frac{4}{3}\left[\alpha_0 - \frac{3\beta^2}{2(4\alpha_0-1)}\right]+O(\beta^3)$$

For arbitrary values of $\beta$, relation (3.6) specifies the equation of a surface in parameter space $(\alpha, \beta, \xi)$. This surface consists of a denumerable number of "pieces" corresponding to the regions $g_n$ of the $(\alpha, \beta)$ plane. The boundaries of these "pieces" in the $(\beta = 0)$ plane are segments of the straight line $\xi = 4\alpha/3$ with $(n-1)^2/4 < \alpha < n^2/4$ $(n = 1, 2, \ldots)$. The surface $\xi = \xi_*(\alpha, \beta)$ was constructed using a computer; the form of this surface is shown in Fig. 2. Calculations show that, within each "piece", the function $\xi = \xi_*(\alpha, \beta)$, for a fixed value of $\beta$, increases monotonically as increases, but as $\beta$ increases the value of $\xi_*$ increases without limit.

We will now consider the interpretation of the results in the parameter space $(\alpha, \beta, d)$. The condition of non-degeneracy is violated when $d_*((\alpha, \beta) = \xi_*((\alpha, \beta)/4 - \alpha$. Calculations show that, for all points $(\alpha, \beta)$ from the regions $g_n$, when $n \geq 2$, we have $d_* < 0$. Therefore, in the regions indicated, the "inverted" top $(d > 0)$ is stable for all values of $d$; the "hanging" top $(d < 0, \alpha > 0)$ is also stable, with the possible exception of the points $d = d_*(\alpha, \beta)$.

In the region $g_1$ (Fig. 3) a curve exists (indicated by the dot-dash curve) on which $d_*(\alpha, \beta) = 0$; this curve emanates from the point (0, 0) and for small $\beta$, is given by the equation $\alpha = 2\beta^2 + O(\beta^3)$; it intersects the boundary curve $\gamma_\xi^{(1)}$ at the point $B(0.103, 0.28)$. For the points $(\alpha, \beta)$ belonging to the curve $d_* = 0$, the stability condition is satisfied for all values $d \neq 0$, i.e. both for the "inverted" and for the "hanging" top. For points of the region $g_1$ situated to the right of the curve $d_* = 0$, we have $d_* < 0$ (the condition of non-degeneracy is violated for the "hanging" top), and for points to the left of this curve we have $d_* > 0$ (the condition of non-degeneracy is violated for the "inverted" top).

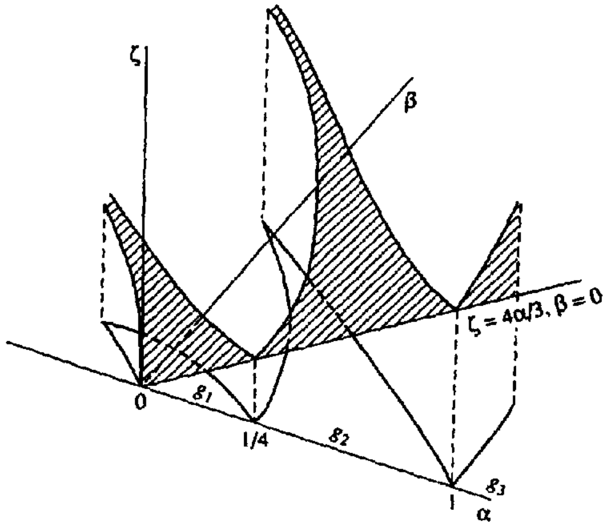


Fig. 2

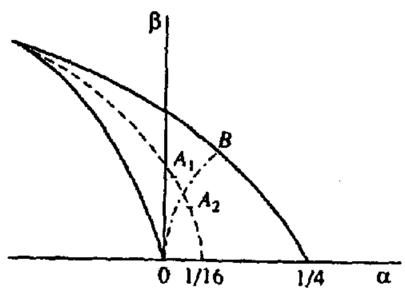


Fig. 3

## 4. STABILITY ON FOURTH-ORDER RESONANCE CURVES

Suppose, now, that the point (α, β) belongs to a fourth-order resonance curve. The non-linear canonical replacement of variables $q_*, p_* \to x, y$ transforms Hamiltonian (3.2) into

$$H = \frac{1}{2}\lambda(x^2+y^2) + \frac{1}{4}c_2(x^2+y^2)^2 + (x_{40}\cos 4\lambda\tau - y_{40}\sin 4\lambda\tau)(x^4 - 6x^2y^2 + y^4) -$$
$$-4(y_{40}\cos 4\lambda\tau + x_{40}\sin 4\lambda\tau)xy(x^2-y^2) + O_6$$

where the coefficient $c_2$ is defined by (3.5), and the constants $x_{40}$ and $y_{40}$ are given by the expressions

$$x_{40} = \int (\chi_1\cos 4\lambda\tau + \chi_2\sin 4\lambda\tau)d\tau, \quad y_{40} = \int(-\chi_1\sin 4\lambda\tau + \chi_2\cos 4\lambda\tau)d\tau$$
$$\chi_1 = -\frac{1}{4}\chi_0[(n_{11}^2+n_{12}^2)^2 - 8n_{11}^2n_{12}^2], \quad \chi_2 = \chi_0 n_{11}n_{12}(n_{11}^2 - n_{12}^2) \tag{4.1}$$
$$\chi_0 = \frac{1}{96\pi}\left(\alpha - \frac{3}{4}\zeta + \beta\cos\tau\right)$$

The equilibrium position $q = 0, p = 0$ of the system with Hamiltonian (1.7) is stable when the condition $|c_2| > 4(x_{40}^2 + y_{40}^2)^{1/2}$ is satisfied, and unstable otherwise [9].

Substituting expressions for $n_{11}$ and $n_{12}$ from (3.1) into (4.1), we obtain

$$x_{40} = \frac{1}{4\varkappa^2}\int[4\mu_1^2\nu_1^2 - (\mu_1^2 - \nu_1^2)\chi_0]d\tau, \quad y_{40} = \frac{1}{\varkappa^2}\int \mu_1\nu_1(\mu_1^2 - \nu_1^2)\chi_0 d\tau$$

Since the integrand in the expression for $y_{40}$ is odd, we have $y_{40} = 0$, and the stability condition will take the form $|c_2| > 4|x_{40}|$. Taking into account (3.1) and Mathieus equation (2.1), we transform the expression for $x_{40}$ into

$$x_{40} = \frac{1}{128\pi\kappa^2}\int [(\mu_1'\nu_1 + \mu_1\nu_1')^2 - (\mu_1\mu_1' - \nu_1\nu_1')^2]d\tau - \frac{\zeta}{512\pi\kappa^2}\int [4\mu_1^2\nu_1^2 - (\mu_1^2 - \nu_1^2)^2]d\tau \qquad (4.2)$$

The boundary of the stability region is defined by the equation $c_2 = +4x_{40}$. The equation $c_2 = 4x_{40}$, taking (3.5) and (4.3) into account is transformed into

$$\zeta = \zeta_1(\alpha, \beta) = 4\frac{\int [(\mu_1'\nu_1 + \mu_1\nu_1')^2 + (\mu_1\mu_1' + \nu_1\nu_1')^2]d\tau}{\int (\mu_1^4 + \nu_1^4 + 6\mu_1^2\nu_1^2)d\tau} \qquad (4.3)$$

and the equation $c_2 = 4x_{40}$ becomes

$$\zeta = \zeta_2(\alpha, \beta) = 4\frac{\int (\mu_1^2\mu_1'^2 + \nu_1^2\nu_1'^2)d\tau}{\int (\mu_1^4 + \nu_1^4)d\tau} \qquad (4.4)$$

Relations (4.3) and (4.4), on the resonance surfaces defined in parameter space ($\alpha$, $\beta$, $\zeta$) by the equations $4\lambda = 2k - 1$ ($k = 1, 2, \ldots$) specify space curves, on passing through which the nature of the stability of the equilibrium position considered changes.

For small values of $\beta$, the quantities $\zeta_1$ and $\zeta_2$ have the form

$$\zeta_{1,2} = \frac{4}{3}\left[\alpha_0 - \frac{3\beta^2}{2(4\alpha_0 - 1)}\right] + O(\beta^3), \quad \alpha_0 = \frac{(2k-1)^2}{16} \quad (k = 1, 2, \ldots)$$

Their principal parts (to terms of the order of $\beta^2$ inclusive) are identical.

The curves $\zeta_1$ ($\alpha$, $\beta$) and $\zeta_2(\alpha, \beta)$ emanate from the common limit point ($\alpha_0$, 0, $4\alpha_0/3$) where they have a common tangent. For all remaining points ($\alpha$, $\beta$) ($\beta \neq 0$) of the resonance curves, as shown by calculations, $\zeta_1 < \zeta_2$. The qualitative form of the curves $\zeta = \zeta_1(\alpha, \beta)$ and $\zeta = \zeta_2(\alpha, \beta)$ is shown in Fig. 4 for one of the resonance surfaces. As $\beta$ increases, the difference $\zeta_2 - \zeta_1$ decreases, and for large values of $\beta$ the curves $\zeta = \zeta_1(\alpha, \beta)$ and $\zeta = \zeta_2(\alpha, \beta)$ converge without limit.

Suppose the point ($\alpha$, $\beta$, $\zeta$) belongs to a resonance surface. If $\zeta < \zeta_1(\alpha, \beta)$ or $\zeta > \zeta_2(\alpha, \beta)$ in this case, we have $|c_2| > 4|x_{40}|$, and the stability condition is satisfied; if, however, $\zeta_1(\alpha, \beta) < \zeta < \zeta_2(\alpha, \beta)$,

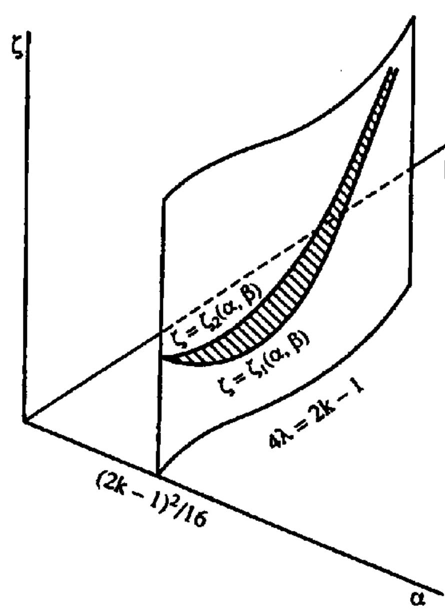


Fig. 4

the opposite inequality holds and we have instability. The instability region on the resonance surface in Fig. 4 is shaded, and outside this region we have stability.

The quantities $d_i = \zeta_i/4 - \alpha$ $(i = 1, 2)$ corresponding to $\zeta_1$ and $\zeta_2$ for the points $(\alpha, \beta)$ of the resonance curves lying in the regions $g_n$ $(n \geq 2)$ are negative for all $\alpha$ and $\beta$, and therefore an instability region $d_1 < d < d_2$ exists only for the "hanging" top (at those points where $\alpha > 0$); the "inverted" top on these resonance curves is always stable.

For the resonance curve $4\lambda = 1$ in the region $g_1$ (Fig. 2, the dashed curve), we have $d_1 = 0$ at the point $A_1(0.036, 0.205)$ and $d_2 = 0$ at the point $A_2(0.042, 0.175)$. For points of the curve $4\lambda = 1$ to the left and above the point $A_1$, we have $d_1 > 0$, $d_2 > 0$, and an instability region $d_1 < d < d_2$ exists for the "inverted" top, while the "hanging" top is stable. For points of the curve $4\lambda = 1$ to the right and below the point $A_2$, we have $d_1 < 0$, $d_2 < 0$, and an instability region exists for the "hanging" top, while the "inverted" top is stable. Finally, for points of the resonance curve lying between the points $A_1$ and $A_2$, $d_1 < 0$, $d_2 > 0$ and an instability region exists both for the "hanging" top $(d_1 < d < 0)$ and for the "inverted" top $(0 < d < d_2)$.

## 5. INVESTIGATION OF STABILITY ON THE BOUNDARY CURVES

We will now consider the case where the parameters $\alpha$ and $\beta$ belong to boundary curves. Using the linear, real, canonical replacement $q, p \to q_*, p_*$, given by the relation

$$\| qp \|^T = \mathbf{N}(\tau) \| q_* p_* \|^T, \quad \mathbf{N}(\tau) = \| n_{ij}(\tau) \| \tag{5.1}$$

Hamiltonian (1.7) is reduced to the form

$$H = \frac{1}{2}\delta p_*^2 - \frac{1}{24}\left(\alpha - \frac{3}{4}\zeta + \beta\cos\tau\right)(n_{11}q_* + n_{12}p_*)^4 + O_6 \tag{5.2}$$

where $\delta$ is equal to 1 or to $-1$, and, along each boundary curve, retains a constant value.

The simplex matrix in (5.1) is $2\pi$-periodic in $\tau$ for points of the curves $\gamma_C^{(2k)}$ and $\gamma_S^{(2k)}$ (where first-order resonance occurs) and $4\pi$-periodic in $\tau$ for points of the curves $\gamma_S^{(2k-1)}$ and $\gamma_C^{(2k-1)}$ (with second-order resonance). The explicit form of this matrix was indicated earlier [2]. For subsequent investigation, only the function $n_{11}(\tau)$ will be required. We will give its form and also the value of the constant $\delta$ for points of each of the boundary curves [2]:

for points of the curves $\gamma_C^{(2k)}$ $(k = 0, 1, 2, \ldots)$

$$\delta = 1, \quad n_{11} = bx_{11}(\tau), \quad b = (|x_{12}(2\pi)|/(2\pi))^{1/2} \tag{5.3}$$

for points of the curves $\gamma_S^{(2k)}$ $(k = 1, 2, \ldots)$

$$\delta = -1, \quad n_{11} = -cx_{12}(\tau), \quad c = (|x_{21}(2\pi)|/(2\pi))^{1/2} \tag{5.4}$$

for points of the curves $\gamma_S^{(2k-1)}$ and $\gamma_C^{(2k-1)}$ $(k = 1, 2, \ldots)$, the values of $\delta$ are equal to 1 and $-1$ respectively, and the functions $n_{11}(\tau)$ are obtained from formulae (5.4) and (5.3) respectively in which $2\pi$ is replaced by $4\pi$.

In relations (5.3) and (5.4) and analogous relations for the curves $\gamma_S^{(2k-1)}$ and $\gamma_C^{(2k-1)}$, the functions $x_{ij}(\tau)$ are elements of the fundamental matrices of solutions of Mathieu's equation (2.1) on the boundary curves (see Section 2).

A non-linear canonical replacement of variables $q_*, p_* \to x, y$ ($2\pi$-periodic in $\tau$ for the curves $\gamma_C^{(2k)}$ and $\gamma_S^{(2k)}$, and $4\pi$-periodic in $\tau$ for the curves $\gamma_S^{(2k-1)}$ and $\gamma_C^{(2k-1)}$) transforms Hamiltonian (5.2) into the following normal form [10]

$$H = \frac{1}{2}\delta y^2 + a_4 x^4 + O_6 \tag{5.5}$$

where $a_4$ is a constant coefficient. If $a_4\delta > 0$, the solution $q = 0, p = 0$ of the system with Hamiltonian (1.7) is stable, but if $a_4\delta < 0$ it is unstable [10].

The coefficient $a_4$ is given by the relation

$$a_4 = -\frac{1}{48\pi s}\int_0^{2\pi s}\left(\alpha - \frac{3}{4}\zeta + \beta\cos\tau\right)n_{11}^4 d\tau \tag{5.6}$$

where $s = 1$ for the curves $\gamma_C^{(2k)}$ and $\gamma_S^{(2k)}$ and $s = 2$ for the curves $\gamma_S^{(2k-1)}$ and $\gamma_C^{(2k-1)}$.

Taking into account that the function $n_{11}(\tau)$ in (5.6) satisfies Mathieu's equation (2.1), we transform the expression for $a_4$ into

$$a_4 = -\frac{1}{16\pi s}\int_0^{2\pi s} n_{11}^2 n_{11}'^2 d\tau + \frac{\zeta}{64\pi s}\int_0^{2\pi s} n_{11}^4 d\tau$$

Whereas the sign of $\delta$ of each boundary curve is retained, the sign of the coefficient $a_4$ at each point $(\alpha, \beta)$ of the boundary curve depends on the value of the parameter $\zeta$: if $\zeta = \zeta_{**}(\alpha, \beta)$, then $a_4 = 0$; if $0 < \zeta < \zeta_{**}$, then $a_4 < 0$; if $\zeta > \zeta_{**}$, then $a_4 > 0$. Here

$$\zeta_{**}(\alpha, \beta) = 4\int_0^{2\pi s} n_{11}^2 n_{11}'^2 d\tau\left(\int_0^{2\pi s} n_{11}^4 d\tau\right)^{-1} \tag{5.7}$$

On the left-hand boundaries of the stability regions (curves $\gamma_C^{(2k)}$ and $\gamma_S^{(2k-1)}$) $\delta = 1$; for points of these curves when $0 < \zeta < \zeta_{**}$ we have $a_4 < 0$, and the solution considered is unstable, but when $\zeta > \zeta_{**}$ we have $a_4\delta > 0$, and the solution is stable. On the right-hand boundaries of the stability regions (curves $\gamma_S^{(2k)}$ and $\gamma_C^{(2k-1)}$) $\delta = -1$, and, conversely, when $0 < \zeta < \zeta_{**}$ we have stability ($a_4\delta > 0$), but when $\zeta > \zeta_{**}$ we have instability ($a_4\delta < 0$) of the solution.

Relation (5.7) specifies the equations of the curves separating the stability and instability regions on the boundary surfaces corresponding in parameter space $(\alpha, \beta, \zeta)$ to the boundary curves considered.

For small values of $\beta$ (small amplitudes of oscillations of the suspension point of the top), (5.7) is transformed as follows:

When $k \neq 1$ for all boundary points $\gamma_C^{(2k)}$, $\gamma_S^{(2k)}$, $\gamma_S^{(2k-1)}$ and $\gamma_C^{(2k-1)}$

$$\zeta_{**} = \frac{4}{3}\left[\alpha_0 - \frac{3\beta^2}{2(4\alpha_0 - 1)}\right] + O(\beta^3)$$

where $\alpha_0$ represents the abscissae of the point on the $\beta = 0$ axis from which the boundary curves emanate ($\alpha_0 = k^2$ for $\gamma_C^{(2k)}$ and $\gamma_S^{(2k)}$, and $\alpha_0 = (2k-1)^2/4$ for $\gamma_S^{(2k-1)}$ and $\gamma_C^{(2k-1)}$);

when $k = 1$

$$\zeta_{**} = \frac{4}{3} - \frac{41}{27}\beta^2 + O(\beta^3) \quad \text{for curve} \quad \gamma_C^{(2)}$$

$$\zeta_{**} = \frac{4}{3} + \frac{5}{27}\beta^2 + O(\beta^3) \quad \text{for curve} \quad \gamma_S^{(2)}$$

$$\zeta_{**} = \frac{1}{3} - \frac{2}{9}\beta + \frac{11}{54}\beta^2 + O(\beta^3) \quad \text{for curve} \quad \gamma_S^{(1)}$$

$$\zeta_{**} = \frac{1}{3} + \frac{2}{9}\beta + \frac{11}{54}\beta^2 + O(\beta^3) \quad \text{for curve} \quad \gamma_C^{(1)}$$

For arbitrary values of $\beta$, the qualitative form of the curves (5.7) is shown in Fig. 5 on two boundary surfaces (corresponding to the boundary curves in the $(\alpha, \beta)$ plane emanating from the common point $(\alpha_0, 0)$ ($\alpha_0 = n^2/4$, $n = 0, 1, 2, \ldots$) of the axis $\beta = 0$). The common limit point (at $\beta = 0$) of these curves has the coordinates $(\alpha_0, 0, 4\alpha_0/3)$; when $\beta$ increases, the functions $\zeta = \zeta_{**}(\alpha, \beta)$ increase without limit.

On the right-hand boundary surface of those shown in Fig. 5 (corresponding to the left-hand boundary of stability in the $(\alpha, \beta)$ plane), the equilibrium position $q = 0, p = 0$ of the system with Hamiltonian (1.7) is stable for points $(\alpha, \beta, \zeta)$ above the curve $\zeta = \zeta_{**}(\alpha, \beta)$, and unstable for points below this curve. On the left-hand boundary surface (corresponding to the right-hand boundary of the stability region in the $(\alpha, \beta)$ plane, on the other hand, for points above the curve $\zeta = \zeta_{**}(\alpha, \beta)$ we have instability, and for points below this curve we have stability. The instability regions on the boundary surfaces in Fig. 5 are shaded.

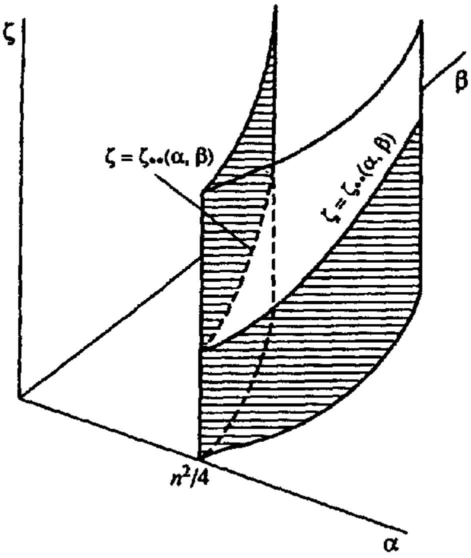


Fig. 5

We will now recalculate the results with respect to parameter $d$. As shown by calculations, the quantity $d_{**}(\alpha, \beta) = \zeta_{**}(\alpha, \beta)/4 - \alpha$ corresponding to $\zeta_{**}(\alpha, \beta)$ is negative for all points $(\alpha, \beta)$ belonging to all boundary curves, except for $\gamma_\zeta^{(0)}$ and $\gamma_\zeta^{(1)}$ (the boundaries of the region $g_1$). For points of the curve $\gamma_\zeta^{(0)}$ (Fig. 3) we have $d_{**} > 0$ (apart from the limit point (0, 0), where $d_{**} = 0$); for points of the curve $\gamma_\zeta^{(1)}$ that lie to the left of the point $B$ (where $d_{**} = 0$) we have $d_{**} > 0$, and for points of this curve to the right of the point $B$ we have $d_{**} < 0$.

Therefore, on all the left-hand boundaries of the regions $g_n$ (apart from $\gamma_\zeta^{(0)}$) the "inverted" top is stable, but on all the right-hand boundaries (apart from $\gamma_\zeta^{(1)}$) it is unstable. On the boundary curve $\gamma_\zeta^{(0)}$ the "inverted" top is unstable when $0 < d < d_{**}$ and stable when $d > d_{**}$. On the curve $\gamma_\zeta^{(1)}$, for points to the right of the point $B$ (Fig. 3) the "inverted" top is unstable; for points of this curve to the left of the point $B$ the top is unstable when $- < d < d_{**}$ and stable when $d > d_{**}$.

The "hanging" top on all the left-hand boundaries of the regions $g_n$ (for points where $\alpha > 0$) is stable when $d_{**} < d < 0$ and unstable when $d < d_{**}$. On the boundary $\gamma_\zeta^{(1)}$ the "hanging" top is stable for points to the left of the point $B$ (when $\alpha > 0$); for points of this curve to the right of the point $B$, and also for all remaining right-hand boundaries of the regions $g_n$ (when $\alpha > 0$) the "hanging" top is stable when $d < d_{**}$ and unstable when $d_{**} < d < 0$.

## 6. COMPARISON WITH THE CLASSICAL RESULT

As indicated in Section 1, the classical "hanging" top is stable in the region of existence $\alpha > 0$, and the "inverted" top is stable when $\alpha \geqslant 0$ and unstable when $\alpha < 0$, In both cases, the points of the positive half-axis $O\alpha$ correspond in Fig. 1 to the stability condition.

When there are oscillations of the suspension point in the $(\alpha, \beta)$ parameter plane, a denumerable number of stability and instability regions are distinguished for both the "hanging" and the "inverted" top. For the "hanging" top these regions lie in the half-plane $\alpha > 0$, and for the "inverted" top each of the stability (and instability) regions, starting in the region of positive values of $\alpha$, intersects the $\alpha = 0$ axis (the boundary of stability for the classical top) and extends without limit towards negative values of $\alpha$ (although the stability regions when $\alpha < 0$ are extremely narrow).

Furthermore, the existence of instability regions on the resonance surfaces (in the space of the three parameters of the problem) and also the diversity of conclusions concerning stability on the boundary surfaces make the picture of stability of a "sleeping" top considerably richer and more complex compared with the classical result.

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